Change of measure for jump processes

Math 622

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Reading material: Shreve Section 11.6

1 Motivation

One of the fundamental concept in Math Finance I regarding the Black-Scholes model is the following: Suppose $S(t)$ satisfies

$$S(t) = S(0) + \int_0^t \mu(u)S(u)du + \int_0^t S(u)dW(u),$$

under the objective probability $\mathbb{P}$. Unless $\mu(u) = r$, the interest rate, (which we supposed to be a constant for simplicty) $e^{-rt}S(t)$ *is not a martingale under* $\mathbb{P}$, and thus we cannot price financial product under $\mathbb{P}$. We need to find another measure $\mathbb{Q}$, the risk neutral measure, so that $e^{-rt}S(t)$ is a martingale udner $\mathbb{Q}$. The key idea is that under $\mathbb{Q}$, it must be the case that $\tilde{W}(t) := \int_0^t(\mu(u) - r)du + W(t)$ is a Brownian motion. So that

$$S(t) = S(0) + \int_0^t rS(u)du + \int_0^t S(u)d\tilde{W}(u)$$

has the right distribution under $\mathbb{Q}$.

Intuitively, the measure $\mathbb{Q}$ is chosen so that we can "modify the drift" of $W(t)$ and still have the new process $\tilde{W}(t)$ being a Brownian motion; which results in modifying the drift of $S(t)$ to the desirable drift(in this case, r).

Now suppose $S(t)$ satisfies

$$S(t) = S(0) + \int_0^t \mu S(u)du + \int_0^t S(u-)dM(u),$$

under some objective probability measure $\mathbb{P}$, where $M(t) = N(t) - \lambda t$ is a compensated Poisson process with rate λ under $\mathbb{P}$. Again, we would like that

$$S(t) = S(0) + \int_0^t rS(u)du + \int_0^t S(u-)d\tilde{M}(u),$$

where $\tilde{M}(t) := M(t) - (r - \mu)t$ is a martingale under a probability measure $\mathbb{Q}$. Again, since $M(t) = N(t) - \lambda t$, it is clear that $\tilde{M}(t)$ is a martingale if $N(t)$ becomes a Poisson process with rate $\lambda - (r - \mu)$ under $\mathbb{Q}$. This note discusses how to choose such a measure $\mathbb{Q}$ for various choices of jump martingales M.

2 Review of change of measure, Girsanov's theorem

Reading material: Shreve's Section 5.2, Ocone's lecture note 2, section 1,2 and 3.

Important points:

(All statements in this section about martingale without qualification will be with respect to the filtration $\mathcal{F}(t)$.)

(i) Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$; $\mathcal{F}(t), 0 \leq t \leq T$ a filtration with $\mathcal{F}(T) = \mathcal{F}$. . If we define another probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}(T))$ via the relation

$$d\mathbb{Q} = Z(T)d\mathbb{P},$$

for some random variable $Z(T)$, that is for all $Y \in \mathcal{F}(T)$

$$\mathbb{E}^{\mathbb{Q}}(Y) := \mathbb{E}^{\mathbb{P}}(Z(T)Y),$$

it must be that $\mathbb{P}(Z(T) \geq 0) = 1$ *and* $\mathbb{E}^{\mathbb{P}}(Z(T)) = 1$.

(ii) (*Restriction of* $\mathbb{Q}$ *to a smaller sigma algebra* $\mathcal{F}(t)$ - Shreve's Lemma 5.2.1) Let $\mathcal{F}(t), 0 \leq t \leq T$ be a filtration associated with a probability space $(\Omega, \mathbb{P}, \mathcal{F}(T))$. If $Z(t)$ is *a* $\mathbb{P}$ *martingale*, $Z(T)$ satisfies the conditions in (i), then for all $Y \in \mathcal{F}(t)$

$$\mathbb{E}^{\mathbb{Q}}(Y) = \mathbb{E}^{\mathbb{P}}(Z(t)Y).$$

Note that this is *not a definition* but a result that follows from the definition in (i) and the fact that Z is a martingale.

(iii) Conditional expectation in change of measure: Suppose the set up in (i) and (ii) apply. Let $Y(t)$ be $\mathcal{F}(t)$ measureable. Then we have for $s \leq t$

$$\mathbb{E}^{\mathbb{Q}}(Y|\mathcal{F}(s)) = \frac{\mathbb{E}^{\mathbb{P}}(Z(t)Y|\mathcal{F}(s))}{Z(s)}.$$

Again note that we need $Z(t)$ *to be a* $\mathbb{P}$*-martingale* here. In general we have to use $\mathbb{E}^{\mathbb{P}}(Z(t)|\mathcal{F}(s))$ in place of $Z(s)$ in the denominator (see Ocone's lecture note 2 Section 2).

(iv) Let $X(t)$ be a $\mathcal{F}(t)$ adapted process, $Z(t)$ *a* $\mathbb{P}$*-martingale* then $X(t)Z(t)$ is a $\mathbb{P}$ martingale if and only if $X(t)$ is *a* $\mathbb{Q}$*-martingale.* (See Ocone's lecture note 2 Section 3)

Think of $X(t)$ as the "Brownian motion with drift" $\tilde{W}(t)$ or the process $\tilde{M}(t)$ in section I. Recall that we want $\tilde{W}(t)$ (or $\tilde{M}(t)$) be a martingale under $\mathbb{Q}$. This statement gives a sufficient condition for this to happen.

3 Some heuristics about Girsanov theorem

3.1 Characterization of Brownian motion

We all know the Levy's characterization of Brownian motion: continuous martingale with quadratic variation on $[0, t]$ equals to t. There is an equivalent characterization:

Theorem 3.1. *Let $X(t)$ be a continuous process such that $X(0) = 0$. Then $X(t)$ is a Brownian motion w.r.t a filtration $\mathcal{F}(t)$ if and only if for all $u \in \mathbb{R}$,*

$$\mathcal{E}^u(X)(t) := \exp(iuX(t) - \frac{1}{2}(iu)^2 t)$$

is a martingale w.r.t $\mathcal{F}(t)$.

We won't give a proof of this theorem. The direction starting with $X(t)$ a Brownian motion is easy. Heuristically, the converse can be argued as followed: if $\mathcal{E}^u(X)(t)$ is a martingale then for $s < t$

$$\mathbb{E}(\exp\big(iu[X(t) - X(s)]\big)|\mathcal{F}(s)) = \exp[\frac{1}{2}(iu)^2(t - s)].$$

Since the RHS is independent of $\mathcal{F}(s)$, $X(t) - X(s)$ is independent of $\mathcal{F}(s)$. Hence it has independent increments. Moreover, the characteristic function of $X(t) - X(s)$ is that of a Normal(0, $t - s$). Hence it has stationary increments, and the increments has Normal(0, $t - s$) distribution. This is the definition of a Brownian motion.

3.2 Choice of $Z(t)$ in Girsanov Theorem

Suppose that $W(t)$ is a $\mathbb{P}$ Brownian motion. We want to find $\mathbb{Q}$ via $d\mathbb{Q} = Z(T)d\mathbb{P}$ so that $\tilde{W}(t) := W(t) + \alpha t$ is a $\mathbb{Q}$ Brownian motion. From the characterization of

Brownian motion, we need $\mathcal{E}^u(\tilde{W})(t)$ to be a $\mathbb{Q}$ martingale. Observe that

$$\exp\big(iu\tilde{W}(t) - \frac{1}{2}(iu)^2 t\big) = \exp\big(iuW(t) - \frac{1}{2}[(iu)^2 - 2iu\alpha]t\big).$$

From point (iv) of Section (2), $Z(t)$ has to be a $\mathbb{P}$-martingale, and $\tilde{W}(t)Z(t)$ also a $\mathbb{P}$-martingale. Since

$$(iu)^2 - 2iu\alpha = (iu - \alpha)^2 - \alpha^2,$$

and clearly

$$\exp\Big((iu + \alpha)W(t) - \frac{1}{2}(iu + \alpha)^2 t\Big)$$

is a $\mathbb{P}$-martingale, we may guess the choice for $Z(t)$ is

$$Z(t) = \exp(-\alpha W(t) - \frac{1}{2}\alpha^2 t).$$

This intution also suggests that if we want $\tilde{W}(t) = W(t) + \int_0^t \alpha(u)du$ to be a $\mathbb{Q}$ Brownian motion, the choice of $Z(t)$ is

$$Z(t) = \exp(\int_0^t \alpha(u)dW(u) - \frac{1}{2}\int_0^t \alpha(u)^2 du),$$

even though the verification now is slightly more involved.

4 Heuristics about change of measure for Poisson process

4.1 Poisson process characterization

Theorem 4.1. *A càdlàg process $N(t)$, $N(0) = 0$, is a Poisson process with rate λ w.r.t $\mathcal{F}(t)$ if and only if for all $u \in \mathbb{R}$*

$$\exp\big(iuN(t) - \lambda t(e^{iu} - 1)\big)$$

is a martingale w.r.t $\mathcal{F}(t)$.

The heuristics for this theorem is similar to the heuristics for the characterization of Brownian motion.

4.2 Choice of $Z(t)$

Suppose $N(t)$ is a Poisson process with rate λ under $\mathbb{P}$. We want to find $\mathbb{Q}$ via $d\mathbb{Q} = Z(T)d\mathbb{P}$ so that $N(t)$ has rate $\tilde{\lambda}$ under $\mathbb{Q}$. By the characterization of Poisson process, we want

$$\exp\big(iuN(t) - \tilde{\lambda}t(e^{iu} - 1)\big)$$

to be a $\mathbb{Q}$-martingale.

Observe that

$$\tilde{\lambda}e^{iu} = \lambda\frac{\tilde{\lambda}}{\lambda}e^{iu} = \lambda e^{i(u - i\log\left(\frac{\tilde{\lambda}}{\lambda}\right))}.$$

Since

$$\exp\left[i(u - i\log\left(\frac{\tilde{\lambda}}{\lambda}\right))N(t) - \lambda t(e^{i(u-i\log\left(\frac{\tilde{\lambda}}{\lambda}\right))} - 1)\right]$$

is a $\mathbb{P}$ martingale, it is clear that we want

$$Z(t) = \exp\left[\log\left(\frac{\tilde{\lambda}}{\lambda}\right)N(t) + (\lambda - \tilde{\lambda})t\right],$$

and it is indeed the case.

5 Change of measures for jump processes

In this section, we give the change of measure results (the choices of $Z(t)$) and the distribution of the process in the new measure for different jump processes. We only summarize the main results. The proofs of these are provided in Shreve's section 11.6.

5.1 Poisson process

Let $N(t)$ be a Poisson process with rate λ under a probabilty $\mathbb{P}$ and $\mathcal{F}(t)$ a filtration for $N(t)$. We want to change the intensity of $N(t)$ via the change of measure: for any $\tilde{\lambda}$, we find a probabilty $\mathbb{Q}$ so that $N(t)$ is a Poisson process with rate $\tilde{\lambda}$ under $\mathbb{Q}$.

Definition 5.1. *Fix $T > 0$. Let $\tilde{\lambda}$ be given. Define*

$$Z(t) := e^{(\lambda - \tilde{\lambda})t}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}, 0 \le t \le T.$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

We have the important results: (Shreve's Lemma 11.6.1, Theorem 11.6.2)

Theorem 5.2. *Z(t) is a $\mathbb{P}$ martingale (w.r.t. $\mathcal{F}(t)$). Under $\mathbb{Q}$, $N(t)$ is a Poisson process with rate $\tilde{\lambda}$.*

5.2 Compound Poisson with discrete jump distribution

Let $Q(t)$ be a compound Poisson process with rate λ (by which we really mean $Q(t) = \sum_{i=1}^{N}(t)Y_i$ and $N(t)$ has rate λ) under a probabilty $\mathbb{P}$ and $\mathcal{F}(t)$ a filtration for $Q(t)$. Recall that each jump of $Q(t)$ has identical distribution Y_i. Here we assume that Y_1 (hence all Y_i's) takes values $y_1, y_2, ..., y_M$ with probability

$$\mathbb{P}(Y_1 = y_m) = p(y_m), 1 \leq m \leq M$$

that is Y_1 has discrete distribution.

We want to change the intensity of $Q(t)$ as well as the distribution of Y_i (that is to change $p(y_m)$) via the change of measure. For any $\tilde{\lambda} > 0$ and $\tilde{p}(y_m) \in (0,1), \sum_{m=1}^{M} \tilde{p}(y_m) = 1$ we find the probabilty $\mathbb{Q}$ so that under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\tilde{\lambda}$ and Y_i has distribution

$$\mathbb{Q}(Y_1 = y_m) = \tilde{p}(y_m), 1 \leq m \leq M.$$

Before we proceed, we state the following Lemma about decomposition of our compound Poisson process (see e.g. Shreve's Corollary 11.3.4 and Ocone's Lecture note 1 section V.D).

Lemma 5.3. *Let $N_m(t)$ denote the number of jumps of $Q(t)$ of size y_m up to and including time t. Then (under $\mathbb{P}$) $N_m(t)$ is a Poisson process with rate $\lambda_m = \lambda p(y_m)$. Moreover, N_m's are independent processes.*

Definition 5.4. *Fix $T > 0$. Let $\tilde{\lambda}_m, m = 1, 2, ..., M$ be given. Define*

$$\begin{aligned} Z_m(t) &:= e^{(\lambda_m - \tilde{\lambda}_m)t}\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}, 0 \leq t \leq T \\ Z(t) &:= \prod_{m=1}^{M} Z_m(t). \end{aligned}$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

We have: (Shreve's Lemma 11.6.4, Theorem 11.6.5)

Theorem 5.5. *Z(t) is a $\mathbb{P}$ martingale (w.r.t. $\mathcal{F}(t)$). Define*

$$\begin{aligned} \tilde{\lambda} &= \sum_{m=1}^{M} \tilde{\lambda}_m \\ \tilde{p}(y_m) &= \frac{\tilde{\lambda}_m}{\tilde{\lambda}}. \end{aligned}$$

Under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\tilde{\lambda}$ and $Q(Y_i = y_m) = \tilde{p}(y_m)$.

We need several remarks to illustrate the important observations here.

Remark 5.6. *First, we mentioned at the beginning of this section that we can choose $\tilde{\lambda}$ and $\tilde{p}(y_m)$, while Theorem (5.5) says we choose $\tilde{\lambda}_m$. The difference is artificial. Indeed, given $\tilde{\lambda}_m$ we can define $\tilde{\lambda}$ and $\tilde{p}(y_m)$ as in the Theorem. But conversely, we can start out with $\tilde{\lambda}$ and $\tilde{p}(y_m)$ and define $\tilde{\lambda}_m := \tilde{p}(y_m)\tilde{\lambda}$. Intuitively it might be easier to remember what we want to accomplish via the Theorem than the details, but it's up to you to decide which form to remember.*

Remark 5.7. *Observe that $\sum_m \lambda_m = \lambda$ and $\sum_m \tilde{\lambda}_m = \tilde{\lambda}$. Moreover, apply Lemma (5.3), we can also decompose $Q(t)$ into M independent Poisson process under $\mathbb{Q}$, each with rate $\tilde{\lambda}_m$. Thus the important observation here is that if we have sums of independent process, then the right change of measure kernel ($Z(t)$) is the product of the the individual change of measure kernels $Z_m(t)$. This philosophy will be repeated in the section for Compound Poisson process and Brownian motion.*

Remark 5.8. *In the same note as the above remark, observe that*

$$Z_m(t) = e^{[\log(\tilde{\lambda}_m) - \log(\lambda)]N_m(t) - (\tilde{\lambda}_m - \lambda)t}.$$

Thus

$$Z(t) = e^{\sum_m \left([\log(\tilde{\lambda}_m) - \log(\lambda_m)]N_m(t) - (\tilde{\lambda}_m - \lambda_m)t \right)}.$$

Intuitively, we have M independent Poisson processes with rate λ_m, and what we are doing here is to change each of them to rate $\tilde{\lambda}_m$ under the probabilty measure $\mathbb{Q}$. The grand effect is Theorem (5.5).

5.3 Compound Poisson with continuous jump distribution

Let $Q(t)$ be a compound Poisson process with rate λ under a probabilty $\mathbb{P}$ and $\mathcal{F}(t)$ a filtration for $Q(t)$. Here we assume Y_i has continuous distribution with density function f.

We want to change the intensity of $Q(t)$ as well as the distribution of Y_i (that is the density f) via the change of measure. For any density function $\tilde{f}$ and $\tilde{\lambda}$, we find a probabilty $\mathbb{Q}$ so that under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\tilde{\lambda}$ and Y_i has continuous distribution with densitry $\tilde{f}$.

There is yet another way to write the process $Z(t)$ in the Section 3.2. Note that

$$\begin{aligned} Z(t) &= e^{(\lambda-\tilde{\lambda})t}\prod_{m=1}^{M}\left(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)}\right)^{N_m(t)} \\ &= e^{(\lambda-\tilde{\lambda})t}\left(\frac{\tilde{\lambda}^{N(t)}\prod_{m=1}^{M}[\tilde{p}(y_m)]^{N_m(t)}}{\lambda^{N(t)}\prod_{m=1}^{M}[p(y_m)]^{N_m(t)}}\right). \end{aligned}$$

Observe that since multiplication is commutative, by rearranging terms,

$$\prod_{m=1}^{M}[p(y_m)]^{N_m(t)} = \prod_{i=1}^{N(t)} p(Y_i).$$

Similarly,

$$\prod_{m=1}^{M}[\tilde{p}(y_m)]^{N_m(t)} = \prod_{i=1}^{N(t)} \tilde{p}(Y_i).$$

Thus

$$\begin{aligned} Z(t) &= e^{(\lambda-\tilde{\lambda})t}\left(\frac{\tilde{\lambda}^{N(t)}\prod_{i=1}^{N(t)}\tilde{p}(Y_i)}{\lambda^{N(t)}\prod_{i=1}^{N(t)}p(Y_i)}\right) \\ &= e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\left(\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}\right). \end{aligned}$$

Intuitively, $p(y_m)$ can be thought of as the "density" function of Y_i's in the discrete setting.This suggests the following when Y_i has continuous distribution.

Definition 5.9. *Fix $T > 0$. Let $\tilde{\lambda} > 0$ and a density function $\tilde{f}$ be given. Define*

$$Z(t) := e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\left(\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right), 0 \le t \le T.$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Remark 5.10. *Since the density function f can be 0, to avoid dividing by 0, we assume $\tilde{f}(y) = 0$ whenver $f(y) = 0$.*

We have the important results: (Shreve's Lemma 11.6.6, Theorem 11.6.7)

Theorem 5.11. *Z(t) is a $\mathbb{P}$ martingale (w.r.t. $\mathcal{F}(t)$). Under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\tilde{\lambda}$ and Y_i has continuous distribution with density $\tilde{f}$.*

5.4 Compound Poisson process and Brownian motion

Let $Q(t)$ be a compound Poisson process with rate λ and $W(t)$ a Brownian motion defined on the *same* probabilty space $(\mathbb{P}, \Omega, \mathcal{F})$ and $\mathcal{F}(t)$ a filtration for $Q(t), W(t)$. Here we also assume Y_i has continuous distribution with density function f.

We want to change the intensity of $Q(t)$, the distribution of Y_i (that is the density f) and the drift of $W(t)$ via the change of measure. More speficially, given a function $\theta(u)$, constant $\tilde{\lambda} > 0$ and density function $\tilde{f}$, we find the probabilty measure $\mathbb{Q}$ such that under $\mathbb{Q}$, $Q(t)$ is compound Poisson with rate $\tilde{\lambda}$, $Y(i)$ has density $\tilde{f}$ and $\tilde{W}(t) := \int_0^t \theta(u)du + W(t)$ is a Brownian motion. Here we also assume that $\tilde{f}(y) = 0$ when $f(y) = 0$.

Remark 5.12. *Before we proceed, we note that necessarily in this case $W(t)$ and $Q(t)$ are independent (see Corollary 11.4.9 and Exercise 11.6 in Shreve's).*

Definition 5.13. *Fix $T > 0$. Let $\tilde{\lambda} > 0$ and a density function $\tilde{f}$ be given. Define*

$$\begin{aligned} Z_1(t) &:= \exp\Big[-\int_0^t \theta(u)dW(u) - \frac{1}{2}\theta^2(u)du\Big]; \\ Z_2(t) &:= e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\left(\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}\right), 0 \le t \le T; \\ Z(t) &:= Z_1(t)Z_2(t). \end{aligned}$$

Also define

$$d\mathbb{Q} = Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T).$$

Remark 5.14. *Note that $Z_1(t)$ is the usual change of measure kernel given by the Girsanov's theorem in Section 5.2. This together with the result in Section* (5.3)*, Remark* (5.7) *and Remark* (5.12)*, it is no surprise that $Z(t)$ has such form.*

We have the important results: (Shreve's Lemma 11.6.8, Theorem 11.6.9)

Theorem 5.15. *Z(t) is a $\mathbb{P}$ martingale (w.r.t. $\mathcal{F}(t)$). Under $\mathbb{Q}$, $Q(t)$ is a compound Poisson process with rate $\tilde{\lambda}$, Y_i has continuous distribution with density $\tilde{f}$, $\tilde{W(t)} = \int_0^t \theta(u)du + W(t)$ is a Brownian motion. Moreover, $Q(t)$ and $\tilde{W(t)}$ are independent under $\mathbb{Q}$.*

Remark 5.16. *Note that we have the parallel between the independence between $Q(t)$ and $W(t)$ under $\mathbb{P}$ and the independence between $Q(t)$ and $\tilde{W}(t)$ under $\mathbb{Q}$. This is important since we do not have any restriction on $\theta(t)$. Indeed $\theta(t)$ can be equal to $Q(t)$ and the independence structure still holds.*

Remark 5.17. *Even though the theorem in Shreve is stated for Y_i having continuous distribution, it is easy to see that a similar result still holds if Y_i has discrete distribution. In this case, under $\mathbb{Q}$, Y_i would also have discrete distribution with a probability distribution $\tilde{p}$ (see Section* (5.2)*). The change of measure kernel $Z_1(t)$ is the same,*

$$Z_2(t) \quad := \quad e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \left(\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)} \right), 0 \le t \le T;$$

and $Z(t) = Z_1(t)Z_2(t)$.

6 Pricing a European call for jump models (1)

6.1 The risk neutral measure

Let $N(t)$ be a Poisson process with rate λ. Supose we model the stock price as

$$dS(t) = \alpha S(t)dt + \sigma S(t-)dM(t),$$

where $M(t) = N(t) - \lambda t$ is a $\mathbb{P}$-martingale. Note that here the only random source of $S(t)$ is from the jump process $N(t)$.

From section 9 of lecture note 1, we also have

$$S(t) = S(0)\exp[(\alpha - \lambda\sigma)t + \log(1+\sigma)N(t)].$$

Let $r > 0$ be the interest rate. We want to find $\mathbb{Q}$ such that $e^{-rt}S(t)$ is a $\mathbb{Q}$ martingale. If that is the case, since

$$dS(t) = rS(t)dt + \sigma S(t-)(dN(t) - [\lambda - \frac{\alpha - r}{\sigma}]dt)$$

clearly we need $N(t)$ to be a Poisson process with rate $\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}$. Since $\tilde{\lambda}$ must be positive, a necessary condition (which implies no arbitrage for the model of $S(t)$) is

$$\lambda - \frac{\alpha - r}{\sigma} > 0.$$

We then define

$$\begin{aligned} d\mathbb{Q} &= Z(T)d\mathbb{P}; \\ Z(t) &= \exp[\log\left(\frac{\tilde{\lambda}}{\lambda}\right) N(t) - (\tilde{\lambda} - \lambda)t)]. \end{aligned}$$

Note that under $\mathbb{Q}$, we write the dynamics of $S(t)$ as

$$dS(t) = rS(t)dt + \sigma S(t-)d\tilde{M}(t),$$

where $\tilde{M}(t) = N(t) - \tilde{\lambda}t$ is a $\mathbb{Q}$-martingale, which is equivalent to

$$S(t) = S(0)\exp[(r - \tilde{\lambda}\sigma)t + \log(1 + \sigma)N(t)].$$

6.2 Pricing of European call

Let $V(t)$ denote the risk-neutral price of a European Call paying $V(T) = (S(T) - K)^+$ at time T. Then by the risk neutral pricing formula, we have

$$V(t) = \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)\big].$$

It remains to find an expression for $V(t)$. Clearly

$$S(T) = S(t)\exp[(r - \tilde{\lambda}\sigma)(T - t) + \log(1 + \sigma)(N(T) - N(t))].$$

So by the Independence Lemma, (Shreve's Lemma (2.3.4)), we only need to evaluate

$$c(t, x) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\big(xe^{(r-\tilde{\lambda}\sigma)(T-t)+\log(1+\sigma)(N(T)-N(t))} - K\big)^+\Big],$$

then we have $V(t) = c(t, S(t))$.

Since $N(T) - N(t)$ has distribution Poisson$(\tilde{\lambda}(T - t))$ under $\mathbb{Q}$, $c(t, x)$ has the expression as an infinite sum, see Shreve's formula (11.7.3). We won't reproduce it here.

7 Pricing a European call for jump models (2)

7.1 Change of measure

Suppose now that

$$dS(t) = \alpha S(t)dt + \sigma S(t-)dM(t),$$

where $M(t) = Q(t) - mt$ is a compensated compound Poisson process under $\mathbb{P}$. Under the risk neutral probability $\mathbb{Q}$,

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t-)d\tilde{M}(t) \\ &= (r - \sigma\tilde{m})S(t)dt + \sigma S(t-)dQ(t). \end{aligned}$$

So clearly we need

(i) $Q(t)$ to be a compound Poisson process under $\mathbb{Q}$ with $\mathbb{E}^{\mathbb{Q}}(Q(1)) = \tilde{m}$.

(ii) $r - \sigma\tilde{m} = \alpha - \sigma m$.

Note that (ii) gives an equation for $\tilde{m}$. If $Q(t) = \sum_{i=1}^{N(t)} Y_i$ and under $\mathbb{Q}$, $N(t)$ is a Poisson process with rate $\tilde{\lambda}$ and $\mathbb{E}(Y_i) = \tilde{\mu}$ then

$$\tilde{m} = \tilde{\lambda}\tilde{\mu}.$$

So (ii) also gives an equation for $\tilde{\lambda}$ and $\tilde{f}$, the distribution of Y_i under $\mathbb{Q}$. From the change of measure sections, we have seen how to choose $Z(T)$ such that the conditions (i) and (ii) are satisfied. Note that this choice may not be unique, as generally equation (ii) has more than 1 unknowns. However, there is also a restriction on the solution $\tilde{\lambda} > 0$. So a simple application of linear algebra result to conclude that there are infinitely many choices of risk neutral measures is not correct.

7.2 Pricing of European call

Observe that

$$dS(t) = (r - \sigma\tilde{m})S(t)dt + \sigma S(t-)dQ(t)$$

has the solution

$$\begin{aligned} S(t) &= S(0)e^{(r-\sigma\tilde{m})t} \prod_{0<s\le t} (1 + \sigma\Delta Q(s)) \\ &= S(0)e^{(r-\sigma\tilde{m})t} \prod_{i=1}^{N(t)} (1 + \sigma Y_i). \end{aligned}$$

Also for $t < T$

$$S(T) = S(t)e^{(r-\sigma\tilde{m})(T-t)} \prod_{i=N(t)+1}^{N(T)} (1 + \sigma Y_i).$$

Observe *the important fact* that $\prod_{i=N(t)+1}^{N(T)}(1+\sigma Y_i)$ *is independent of* $\mathcal{F}(t)$, where $\mathcal{F}(t)$ is a filtration for $Q(t)$. We give an explanation in the next subsection.

Thus $V(t)$, the risk-neutral price of a European Call paying $V(T) = (S(T) - K)^+$ at time T for this model is

$$\begin{aligned} V(t) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S(T) - K)^+ | \mathcal{F}(t)\big] \\ &= c(t, S(t)), \end{aligned}$$

where

$$c(t,x) \ := \ e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[[xe^{(r-\sigma\tilde{m})(T-t)} \prod_{i=N(t)+1}^{N(T)} (1 + \sigma Y_i) - K]^+\Big].$$

Since Y_i are independent of $N(T)-N(t)$, again we can condition on $N(T)-N(t) = j, j = 1, 2, ...$ to get

$$c(t,x) \ = \ e^{-r(T-t)} \sum_{j=0}^{\infty} \kappa(j,x) e^{-\tilde{\lambda}(T-t)} \frac{[\tilde{\lambda}(T-t)]^j}{j!},$$

where

$$\kappa(j,x) = \mathbb{E}^{\mathbb{Q}}\Big[\big(xe^{(r-\sigma\tilde{m})(T-t)} \prod_{i=1}^{j}(1 + \sigma Y_i) - K\big)^+\Big].$$

7.3 The independence of $\prod_{i=N(t)+1}^{N(T)}(1+\sigma Y_i)$ from $\mathcal{F}(t)$

It is because

$$\prod_{i=N(t)+1}^{N(T)} (1 + \sigma Y_i) = \prod_{i=1}^{N(T)-N(t)} (1 + \sigma Y_{i+N(t)}).$$

Since $N(T) - N(t)$ is independent of $\mathcal{F}(t)$, by the Independence lemma,

$$\mathbb{E}\big[\prod_{i=1}^{N(T)-N(t)} (1 + \sigma Y_{i+N(t)}) | \mathcal{F}(t)\big] = f(N(t)),$$

where

$$f(k) = \mathbb{E}\Big[\prod_{i=1}^{N(T)-N(t)} (1+\sigma Y_{i+k})\Big].$$

But since Y_i are indepedent of $N(t)$,

$$\begin{aligned} f(k) &= \mathbb{E}\Big[\mathbb{E}\Big(\prod_{i=1}^{N(T)-N(t)} (1+\sigma Y_{i+k})\Big| N(T)-N(t)\Big)\Big] \\ &= \mathbb{E}\Big[[\mathbb{E}\big(1+\sigma Y_1\big| N(T)-N(t)\big)]^{N(T)-N(t)}\Big] \\ &= \mathbb{E}\Big[\big(\mathbb{E}(1+\sigma Y_1)\big)^{N(T)-N(t)}\Big], \end{aligned}$$

which does not depend on k. Thus

$$\mathbb{E}[\prod_{i=1}^{N(T)-N(t)} (1+\sigma Y_{i+N(t)}) | \mathcal{F}(t)] = \mathbb{E}[\prod_{i=1}^{N(T)-N(t)} (1+\sigma Y_{i+N(t)})],$$

and we get the independence we need.

8 Pricing a European call for jump models (3)

8.1 Change of measure

Suppose now that

$$dS(t) = \alpha S(t)dt + \sigma S(t-)dW(t) + S(t-)dM(t),$$

where $M(t) = Q(t) - mt$ is a compensated compound Poisson process under $\mathbb{P}$. Under the risk neutral probability $\mathbb{Q}$,

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t-)d\tilde{W}(t) + S(t-)d\tilde{M}(t) \\ &= (r-\tilde{m})S(t)dt + \sigma S(t-)d\tilde{W}(t) + S(t-)dQ(t), \end{aligned}$$

where $\tilde{W}(t) := W(t) + \theta t$ is a $\mathbb{Q}$ Brownian motion and $Q(t)$ is compound Poisson with $\mathbb{E}^{\mathbb{Q}}(Q(1)) = \tilde{m}$.

Thus the equation that θ and $\tilde{m}$ have to satisfy is

$$r + \sigma\theta - \tilde{m} = \alpha - m.$$

Solving this equation for θ and $\tilde{m}$ and use the change of measure result discussed above, we can find $\mathbb{Q}$ such that $e^{-rt}S(t)$ is a $\mathbb{Q}$ - martingale.

8.2 Pricing of European call

Observe that

$$dS(t) = (r-\tilde{m})S(t)dt + \sigma S(t-)d\tilde{W}(t) + S(t-)dQ(t),$$

has the solution

$$S(t) = S(0)\exp\Big[(r-\tilde{m}-\frac{1}{2}\sigma^2)t + \sigma\tilde{W}(t)\Big]\prod_{i=1}^{N(t)}(1+Y_i).$$

Hence for $t < T$,

$$S(T) = S(t)\exp\Big[(r-\tilde{m}-\frac{1}{2}\sigma^2)(T-t) + \sigma(\tilde{W}(T)-\tilde{W}(t))\Big]\prod_{i=N(t)+1}^{N(T)}(1+Y_i),$$

where we have the *independence* of $\tilde{W}(T)-\tilde{W}(t)$ and $\prod_{i=N(t)+1}^{N(T)}(1+Y_i)$ with respect to $\mathcal{F}(t)$ and *also with respect to each other.*

Thus $V(t)$, the risk-neutral price of a European Call paying $V(T) = (S(T)-K)^+$ at time T for this model is

$$\begin{aligned} V(t) &= \mathbb{E}^{\mathbb{Q}}\big[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}(t)\big] \\ &= c(t, S(t)), \end{aligned}$$

where

$$c(t,x) := e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\Big[\Big(xe^{(r-\tilde{m}-\frac{1}{2}\sigma^2)(T-t)+\sigma(\tilde{W}(T)-\tilde{W}(t))}\prod_{i=N(t)+1}^{N(T)}(1+Y_i) - K\Big)^+\Big].$$

To find an expression for $c(t,x)$, we first condition on $\prod_{i=N(t)+1}^{N(T)}(1+Y_i)$ and use the independence lemma to define a function $\kappa(t,x)$ as

$$\kappa(t,x) := e^{-rt}\mathbb{E}^{\mathbb{Q}}\Big[\Big(xe^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Y} - K\Big)^+\Big],$$

where Y has standard normal distribution. Note that we have an explicit expression for Y from the Black-Scholes formula. Then

$$c(t,x) = \mathbb{E}^{\mathbb{Q}}\Big[\kappa(T-t, xe^{-\tilde{m}(T-t)}\prod_{i=N(t)+1}^{N(T)}(1+Y_i))\Big].$$

Now again conditioning on $N(T)-N(t) = j$ and using the independence between $Y_i's$ and $N(T)-N(t)$ we have

$$c(t,x) = e^{-\tilde{\lambda}(T-t)}\frac{(\tilde{\lambda}(T-t))^j}{j!}\mathbb{E}^{\mathbb{Q}}\Big[\kappa(T-t, xe^{-\tilde{m}(T-t)}\prod_{i=1}^{j}(1+Y_i))\Big].$$